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PROBABILISTIC REPRESENTATIONS OF THE DENSITY FUNCTION OF THE ASSET PRICE AND OF VANILLA OPTIONS IN LINEAR STOCHASTIC VOLATILITY MODELS

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Abstract. We derive probabilistic representations for the probability density function of the arbitrage price of a financial asset and the price of European call and put options in a linear stochastic volatility model with correlated Brownian noises. In such models the asset price satisfies a linear SDE with coefficient of linearity being the volatility process. Examples of such models are considered, including a log-normal stochastic volatility model. In all examples a closed formula for the density function is given. In the Appendix we present a conditional version of the Donati-Martin and Yor formula.

Key words: stochastic volatility model, probabilistic representation, correlated Brownian motions, density function, log-normal stochastic volatility model, Bessel process, arbitrage price, vanilla option

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1. Introduction

The famous Black-Scholes model with its relatively stringent assumptions does not capture many phenomena of modern financial markets. A prime example is the stochastic nature of the financial asset's volatility, called volatility smile (see for example Hull and White [6]). In recent years many stochastic volatility models have been introduced and developed, but as was to be expected, making the volatility stochastic has complicated the models considerably (see for example Rebonato [16]). It is not our aim to review the broad range of stochastic volatility models but rather to focus on and develop the idea of modeling stochastic volatility in the simplest possible but effective way. SABR is an excellent example of a model complex in nature but simple in form. This well known and celebrated model, introduced in 2002 by Hagan et al. [5], has since been effectively used and investigated by market practitioners. Soon after its introduction it turned out to be more effective than Black-Scholes and local volatility models. The key idea in SABR is to make stochastic volatility a simple stochastic process and then shift the difficulty of finding the financial asset's distribution to the level of finding the distribution of

the diffusion describing the asset price. For SABR, determining closed formulae for the asset distribution remains, in general, an unsolved task (as far as the authors know).

The task of determining closed formulae for the probability distribution for SABR with parameter beta equal to one, called a log-normal stochastic volatility model, has recently been investigated by Maghsoodi [11], [12]. In this case it is not particularly difficult to write out the solution of the model, i.e. the stochastic process representing the asset price, as the exponential of a linear combination of a pair of correlated Brownian motion functionals. Maghsoodi used the techniques of changing time and measure to find the joint density function of these functionals. The same techniques had been used earlier by Yor in the problem of valuation of Asian options (see [20]). However, Maghsoodi has not mentioned that the asset price loses the martingale property in a log-normal stochastic volatility model in the case of positive correlation between the asset and its volatility.

In our work we propose a kind of reversing the idea of the SABR model and continue the line of research of Hull and White [6] followed also by Romano and Touzi [18] as well as by Leblanc [10]. We shift the complicated nature of the model to the level of the process representing volatility, keeping the diffusion of the asset relatively simple. So, we assume that the asset price process X satisfies $dX_t = Y_tX_tdW_t$ with Y given by $dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dZ_t$, where the processes W, Z are correlated Brownian motions. We call this model a linear stochastic volatility model. Under some natural assumptions we are able to prove that the distribution of the asset has a density function and derive the probabilistic representation of that function (Theorem 2.2). This representation depends on some functionals of the process representing volatility, so the problem of determining the asset distribution reduces to finding the distribution of a 2-dimensional functional of the volatility.

In Section 3, we point out two nontrivial examples of such models in which we can benefit from probabilistic representations of the asset density function. The first example is a log-normal stochastic volatility model which is a SABR model with beta equal to one. Using the result of Matsumuoto and Yor [14] who derived the density function for the vector of Brownian motion with drift and its exponential functional, we find closed formulae for the density function in a stochastic log-normal volatility model. The second example we present is the one with volatility being a 3-dimensional Bessel process (briefly, a BES(3)) starting from 1. Following the ideas of Pal and Protter [15] as well as Donati-Martin and Yor [3], we use change of measure techniques and obtain the Laplace transforms for the volatility functionals. Next, we derive closed formulae for the density function in this model, using Theorem 2.2.

In Section 4 we derive probabilistic representations for European call and put option prices in linear stochastic volatility models. The probabilistic representation for vanilla option prices is independent of the distribution of the asset price itself. This allows us to obtain formulae for the arbitrage prices of vanilla options in a log-normal stochastic volatility model.

Similar probabilistic representations for European call and put option arbitrage prices in a linear stochastic volatility model have also been given by Romano and Touzi [18], but in a slightly different context. They considered a slightly different model and established a set of assumptions under which they obtained probabilistic representation results while proving the convexity of European call and put options

in their setting (also linear in our sense). In particular, they assumed that the coefficients μ and σ in the definition of Y are bounded. In our work we relax this assumption (see Theorem 4.1). In our examples the drift coefficient is not bounded, but the probabilistic representation for option prices holds. It is easy to check that in the case of the volatility process Y being BES(3) with $Y_0 = 1$ the condition $\rho \leq 0$ implies that X is a martingale, and the probabilistic representation works well in that case. The same is true in the case of a log-normal stochastic volatility model. Thus for linear stochastic volatility models our result is more general than the one in [18]. It should be mentioned that Leblanc [10] gives the arbitrage price of call option in a linear stochastic volatility model, with some concrete examples of volatility, in terms of Laplace and Fourier transforms.

Closed formulae for the density function and vanilla option prices in a stochastic log-normal volatility model are interesting and important for applications since such models are popular, especially among forex exchange options traders (see [5]). Similar results for log-normal stochastic volatility models were also presented in [11] and [12]. It should be mentioned that during the writing of this text we discovered that Forde [4] also found the link between the result of Matsumuoto and Yor [14] and SABR for beta equal to one, but he did it only for uncorrelated price and volatility. Thus our result for a log-normal stochastic volatility model is deeper than that in [4] as it requires only the asset price be a martingale.

In the Appendix we give a conditional version of the Donati-Martin and Yor formula [3] and establish the density function of the vector $(B_t, \int_0^t B_u^2 du)$, where B is a standard Brownian motion.

2. Probabilistic representation of the density function of the asset price in a linear stochastic volatility model

2.1. **A linear stochastic volatility model.** We consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T < \infty$, satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Without loss of generality we assume the savings account to be constant and identically equal to one. Moreover, we assume that the price X_t of the underlying asset at time t has a stochastic volatility Y_t, and the dynamics of the vector (X, Y) is given by

$$dX_t = Y_t X_t dW_t, \tag{1}$$

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dZ_t, \tag{2}$$

where X_0, Y_0 are positive constants, the processes W, Z are correlated Brownian motions, $d\langle W, Z\rangle_t = \rho dt$ with $\rho \in (-1, 1)$, and $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous functions such that there exists a unique strong solution of (2), which is positive and $\int_0^T Y_u^2 du < \infty$ $\mathbb{P}$-a.s.

Under these assumptions the process X has the form

$$X_t = X_0 e^{\int_0^t Y_u dW_u - \int_0^t Y_u^2 du/2}, \tag{3}$$

and this is a unique strong solution of SDE (1) on $[0, T]$. The existence and uniqueness follow directly from the assumptions on Y_t and the well known properties of stochastic exponent (see, e.g., Revuz and Yor [17]). The process X is a local martingale, so there is no arbitrage on the market so defined.

We call this model a *linear stochastic volatility model*, because the SDE (1) governing the asset price is linear with respect to the asset itself with coefficient being the stochastic volatility Y.

Remark 2.1. a) It is worth mentioning that the constant ρ in the model can be replaced by a measurable, deterministic function $\rho : [0,T] \to (-1,1)$ and the results of this work remain true with minor modifications.
b) Our standing assumption is $|\rho| < 1$. However, our methods allow finding the distribution of X_t in the case $\rho = \pm 1$ (see Remark 2.4).

2.2. **Existence of the density function and its probabilistic representation.** We start with the main theorem of the paper on existence of the density function of the underlying asset in a linear stochastic volatility model, and its probabilistic representation. This representation allows us to find a closed formula for the density function (see examples in the next section), which is important for applications (see, e.g., Carmona and Durrleman [2]).

Theorem 2.2. *Fix $t \in [0,T]$. If*

$$\mathbb{E}\int_0^t Y_u^{-2} du < \infty, \tag{4}$$

then the random variable X_t has a continuous density function g_{X_t}. Moreover, the density function has the probabilistic representation

$$g_{X_t}(r) = \mathbb{E}\Big[\frac{1}{r\sigma_Z(t)}\Phi'\Big(\frac{\ln\frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\Big)\Big], \tag{5}$$

where Φ is the cumulative distribution function of a standard Gaussian random variable $N(0,1)$, and

$$\mu_Z(t) = \rho\int_0^t Y_u dZ_u - \frac{1}{2}\int_0^t Y_u^2 du, \tag{6}$$

$$\sigma_Z^2(t) = (1-\rho^2)\int_0^t Y_u^2 du. \tag{7}$$

Proof. Notice that we can represent W in the form

$$W_t = \rho Z_t + \sqrt{1-\rho^2} B_t, \tag{8}$$

where (B,Z) is the standard two-dimensional Wiener process.

Step 1. Proof of the existence of density. We prove that the characteristic function ϕ_{U_t} of $U_t := \ln X_t$ is integrable. The Itô lemma applied to (1) together with (2) and (8) implies that

$$\ln X_t = \ln X_0 + \theta_Z(t) + \theta_B(t),$$

where

$$\theta_Z(t) := \rho\int_0^t Y_u dZ_u - \frac{1}{2}\rho^2\int_0^t Y_u^2 du,$$

$$\theta_B(t) := \sqrt{1-\rho^2}\int_0^t Y_u dB_u - \frac{1}{2}(1-\rho^2)\int_0^t Y_u^2 du.$$

Let $\mathcal{F}_t^Z = \sigma(Z_u : u \le t)$. For $s \in \mathbb{R}$,

$$|\phi_{U_t}(s)| = \big|\mathbb{E}\big(e^{is\theta_Z(t)}\mathbb{E}[e^{is\theta_B(t)}|\mathcal{F}_t^Z]\big)\big| \le \mathbb{E}\big|\mathbb{E}[e^{is\theta_B(t)}|\mathcal{F}_t^Z]\big|.$$

Since SDE (2) has the unique strong solution, there exists an appropriately measurable function $\Psi(\ , \ )$ such that $Y = \Psi(Y_0, Z)$ (see e.g. [7]). Together with the fact that the processes B and Z are independent Brownian motions, this implies that the random variable $\theta_B(t)$, for a fixed trajectory of Z_u, $u \le t$, has Gaussian distribution with mean

$$\hat{\mu} = -\frac{1}{2}(1-\rho^2)\int_0^t Y_u^2 du$$

and variance

$$\hat{\sigma}^2 = (1-\rho^2)\int_0^t Y_u^2 du.$$

Consequently, $\mathbb{P}$-a.s.,

$$\mathbb{E}[e^{is\theta_B(t)}|\mathcal{F}_t^Z] = e^{i\hat{\mu}s - \hat{\sigma}^2 s^2/2},$$

and hence

$$\mathbb{E}\Big|\mathbb{E}[e^{is\theta_B(t)}|\mathcal{F}_t^Z]\Big| = \mathbb{E}e^{-s^2(1-\rho^2)\int_0^t Y_u^2 du/2}.$$

Therefore, by the Fubini theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|\phi_{U_t}(s)|ds &\le \int_{-\infty}^{\infty}\mathbb{E}e^{-s^2(1-\rho^2)\int_0^t Y_u^2 du/2}ds = \mathbb{E}\int_{-\infty}^{\infty}e^{-s^2(1-\rho^2)\int_0^t Y_u^2 du/2}ds \\ &= \mathbb{E}\sqrt{\frac{2\pi}{(1-\rho^2)}\Big(\int_0^t Y_u^2 du\Big)^{-1}} \le \sqrt{\frac{2\pi}{(1-\rho^2)}\mathbb{E}\Big(\int_0^t Y_u^2 du\Big)^{-1}} = I.\end{aligned}$$

In the last inequality we have used the Jensen inequality for the concave function $f(x) = \sqrt{x}$. Now, using the integral Jensen inequality for the convex function $f(x) = 1/x$, for $x > 0$, we have

$$\frac{1}{\int_0^t Y_u^2 du} \le \frac{1}{t^2}\int_0^t Y_u^{-2} du. \tag{9}$$

Hence and by assumption (4),

$$I = \sqrt{\frac{2\pi}{(1-\rho^2)}\mathbb{E}\Big(\int_0^t Y_u^2 du\Big)^{-1}} \le \sqrt{\frac{2\pi}{(1-\rho^2)}}\frac{1}{t}\sqrt{\mathbb{E}\int_0^t Y_u^{-2} du} < \infty.$$

In such a way we have proved that the characteristic function ϕ_{U_t} is integrable. By the Inverse Fourier Transform theorem there exists a bounded and continuous density function of U_t. Since $X_t = e^{U_t}$, we conclude that there exists a continuous density function of X_t.

Step 2. Proof of (5)*—probabilistic representation of* X_t. For fixed $r > 0$, using the same arguments as in Step 1 we have, by (3) and (8),

$$\begin{aligned}\mathbb{P}(X_t \le r) &= \mathbb{E}1_{\left\{X_0 \exp\left(\int_0^t Y_u dW_u - \frac{1}{2}\int_0^t Y_u^2 du\right) \le r\right\}} \\ &= \mathbb{E}\mathbb{E}\Big[1_{\left\{\rho\int_0^t Y_u dZ_u + \sqrt{1-\rho^2}\int_0^t Y_u dB_u - \frac{1}{2}\int_0^t Y_u^2 du \le \ln\frac{r}{X_0}\right\}}\Big|\mathcal{F}_t^Z\Big] \\ &= \mathbb{E}\mathbb{P}\Big(\mu_Z(t) + \sigma_Z(t)g \le \ln\frac{r}{X_0}\Big|\mathcal{F}_t^Z\Big) = \mathbb{E}\mathbb{P}\Big(g \le \frac{\ln\frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\Big|\mathcal{F}_t^Z\Big) \\ &= \mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\Big) := h(r), \end{aligned} \tag{10}$$

where g is a standard Gaussian random variable independent of $\mathcal{F}_t^Z$, $\mu_Z(t)$ and $\sigma_Z^2(t)$ are given by (6) and (7), respectively. The function h, defined for $r > 0$ by (10), is continuous. Moreover,

$$\frac{\partial}{\partial r}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big) = \frac{1}{r\sigma_Z(t)}\Phi'\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)$$
$$\le \frac{1}{\sqrt{2\pi}r\sigma_Z(t)} \le \frac{1}{\sqrt{2\pi}r}\Big[1+\frac{1}{\sigma_Z^2(t)}\Big] = \frac{1}{\sqrt{2\pi}r}\Big[1+\frac{1}{(1-\rho^2)\int_0^t Y_u^2du}\Big] \le I_1,$$

where

$$I_1 = \frac{1}{\sqrt{2\pi}r}\Big[1+\frac{1}{(1-\rho^2)}\frac{1}{t^2}\int_0^t Y_u^{-2}du\Big].$$

In the last inequality we use (9). From assumption (4) it follows that the random variable I_1 is integrable. Consequently,

$$h_t'(r) = \frac{\partial}{\partial r}\mathbb{P}(X_t\le r) = \mathbb{E}\Big[\frac{1}{r\sigma_Z(t)}\Phi'\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)\Big]. \tag{11}$$

Now we prove that h_t' is a density. It suffices to show that $\int_0^\infty h_t'(r)dr = 1$. Using the Lebesgue dominated convergence theorem we get

$$\lim_{r\to 0}\mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big) = \mathbb{E}\Big[\lim_{r\to 0}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)\Big] = 0. \tag{12}$$

Hence, by (11) and the Fubini theorem,

$$\int_0^\infty h_t'(r)dr = \mathbb{E}\int_0^\infty\Big[\frac{1}{r\sigma_Z(t)}\Phi'\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)\Big]dr$$
$$= \mathbb{E}\Big[\int_0^\infty\frac{\partial}{\partial r}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)dr\Big] = \mathbb{E}\Big[\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)\Big|_0^\infty\Big] = 1,$$

so $g_{X_t}(r) = h_t'(r)$ is the density function of X_t. □

Remark 2.3. From the last theorem it is clear that finding the distribution of X_t, for fixed t, reduces to deriving the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$, provided (4) is satisfied.

Remark 2.4. Note that in the case $\rho = \pm 1$ we cannot write the formula (5) for the density of X_t. However, we have $W = \pm Z$ in this case and

$$X_t = X_0 e^{\pm\int_0^t Y_u dZ_u - \int_0^t Y_u^2 du/2}, \tag{13}$$

so the problem of finding the distribution of X_t, for fixed t, reduces, as previously, to deriving the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$. In the case of a lognormal stochastic volatility model we can use the results of Matsumoto and Yor [14] to obtain the distribution of $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$, as we can express its components in terms of A_t and V_t just as in the proof of Theorem 3.1 and use (19). In the case of volatility being BES(3), Theorem 3.3 gives us the Laplace transform of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$, so after inverting it we obtain its distribution.

3. Application of the probabilistic representation

Using the probabilistic representation of the density, we can find a closed form of the density function in some cases including trivially the Black-Scholes model, and nontrivially a log-normal stochastic volatility model and a linear stochastic volatility model with volatility being a 3-dimensional Bessel process.

3.1. **Closed form of the density function in Black-Scholes and log-normal stochastic volatility models.** The Black-Scholes model and log-normal stochastic volatility models are special cases of linear stochastic volatility models. A log-normal model was considered by Hull and White in the case of uncorrelated noises [6], and it is a SABR model with $\beta = 1$, introduced in 2002 by Hagan et al. [5], in the case of correlated noises.

The case of the Black-Scholes model is trivial. The process $Y_t \equiv \sigma > 0$ represents the volatility of the asset, $\rho = 0$, so it is a linear stochastic volatility model and Theorem 2.2 gives the well known density function of a random variable with log-normal distribution.

A log-normal stochastic volatility model is much less trivial. In this case the functions appearing in the SDE for volatility are $\mu(y) \equiv 0$ and $\sigma(y) = \sigma y$ for $y > 0$, where σ is a positive constant. Thus the process Y is a geometric Brownian motion and

$$Y_t = Y_0 e^{\sigma Z_t - \sigma^2 t/2}. \tag{14}$$

Moreover, it is easy to check that for any $t \leq T$,

$$\mathbb{E} \int_0^t Y_u^2 du = Y_0^2 \frac{1}{\sigma^2}[e^{\sigma^2 t} - 1] < \infty,$$

so a log-normal stochastic volatility model belongs to the class of linear stochastic volatility models.

Our main goal in this subsection is to find, for a log-normal stochastic volatility model, a closed form of the density function of the random variable X_t for fixed nonnegative t (see [12] for another result in this direction). We determine the true distribution of the price process, so we find a simple way to price derivatives in that model.

Theorem 3.1. *In a log-normal stochastic volatility model the density function of the price X_t of the underlying asset has the form*

$$g_{X_t}(r) = \int_{-\infty}^{\infty} \int_0^{\infty} \left[\frac{1}{rY_0 \sqrt{y \frac{1-\rho^2}{\sigma^2}}} \Phi' \left(\frac{\ln \frac{r}{X_0} - f(x,y) + Y_0^2 y \frac{1-\rho^2}{\sigma^2}}{Y_0 \sqrt{y \frac{1-\rho^2}{\sigma^2}}} \right) \right] G_{t\sigma^2}(x,y) dy dx,$$

where

$$f(x,y) = \frac{\rho}{\sigma} Y_0 [e^x - 1] - \frac{\rho^2}{2\sigma^2} Y_0^2 y, \tag{15}$$

$$G_t(x,y) = \exp\left(-\frac{x}{2} - \frac{t}{8} - \frac{1+e^{2x}}{2y} \right) \theta\left(\frac{e^x}{y}, t \right) \frac{1}{y}, \tag{16}$$

and the function θ is defined, using hyperbolic functions, by the formula

$$\theta(r,t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\pi^2/2t} \int_0^\infty e^{-\xi^2/2t - r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi. \tag{17}$$

Proof. Set $\tilde{Y}_t := Y_{t/\sigma^2}$. It is clear, from (14), that

$$\tilde{Y}_t = Y_0 e^{-t/2 + \tilde{Z}_t},$$

where $\tilde{Z}_t = \sigma Z_{t/\sigma^2}$ is a Brownian motion. We can express $\mu_Z(t)$ and $\sigma_Z^2(t)$, defined by (6) and (7), in terms of $\tilde{Y}_t$:

$$\mu_Z(t) = \frac{\rho}{\sigma}[\tilde{Y}_{t\sigma^2} - \tilde{Y}_0] - \frac{1}{2\sigma^2}\int_0^{t\sigma^2} \tilde{Y}_u^2 du, \quad \sigma_Z^2(t) = \frac{1-\rho^2}{\sigma^2}\int_0^{t\sigma^2} \tilde{Y}_u^2 du.$$

Let

$$V_t := \tilde{Z}_t - \frac{t}{2}, \qquad A_t := \int_0^t e^{2V_s} ds.$$

Then $\tilde{Y}_t = Y_0 e^{V_t}$ and $\int_0^t \tilde{Y}_u^2 du = Y_0^2 A_t$. Since

$$\mathbb{E}\int_0^t Y_u^{-2} du = \frac{1}{3Y_0\sigma^2}\left(e^{3\sigma^2 t} - 1\right) < \infty,$$

we can use Theorem 2.2 and write the density function $g_{X_{t/\sigma^2}}$ in terms of V_t and A_t:

$$g_{X_{t/\sigma^2}}(r) = \mathbb{E}\left[\frac{1}{rY_0\sqrt{A_t\frac{1-\rho^2}{\sigma^2}}}\Phi'\left(\frac{\ln\frac{r}{X_0} - f(V_t, A_t) + Y_0^2 A_t \frac{1-\rho^2}{\sigma^2}}{Y_0\sqrt{A_t\frac{1-\rho^2}{\sigma^2}}}\right)\right], \tag{18}$$

where f is given by (15). Now, we use the result of Matsumoto and Yor [14] which gives the density function of the vector (V_t, A_t): they proved that for $t > 0$, $y > 0$ and $x \in \mathbb{R}$,

$$\mathbb{P}(V_t \in dx, A_t \in dy) = G_t(x,y)dxdy, \tag{19}$$

where

$$G_t(x,y) = \exp\left(-\frac{x}{2} - \frac{t}{8} - \frac{1+e^{2x}}{2y}\right)\theta\left(\frac{e^x}{y}, t\right)\frac{1}{y},$$
$$\theta(r,t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\pi^2/2t}\int_0^\infty e^{-\xi^2/2t - r\cosh(\xi)}\sinh(\xi)\sin\left(\frac{\pi\xi}{t}\right)d\xi.$$

Hence (18) can be written in the form

$$g_{X_{t/\sigma^2}}(r) = \int_{-\infty}^\infty \int_0^\infty \left[\frac{1}{rY_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\Phi'\left(\frac{\ln\frac{r}{X_0} - f(x,y) + Y_0^2 y\frac{1-\rho^2}{\sigma^2}}{Y_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\right)\right]G_t(x,y)dydx,$$

with f, G given by (15) and (16). Replacing t by $t\sigma^2$ in the above formula finishes the proof. □

Remark 3.2. Although the formula for the density function of the price in the log-normal stochastic volatility model is complicated, this result describes the true, not approximate, probabilistic law for X_t. If X is a martingale, so describes the arbitrage price of the asset, having the density function we are able to use the risk-neutral valuation formula to price attainable European contingent claims. For

example, evaluating the arbitrage price of power option (see, e.g., [19]) reduces, by Theorem 3.1, to computing the integral

$$\int_0^\infty \int_{-\infty}^\infty \int_0^\infty \frac{[(r-K)^+]^\alpha}{rY_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\Phi'\Big(\frac{\ln\frac{r}{X_0} - f(x,y) + Y_0^2 y\frac{1-\rho^2}{\sigma^2}}{Y_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\Big)G_{T\sigma^2}(x,y)dydxdr,$$

with f, G given by (15) and (16). We stress that in this way we reduce the valuation problem to numerical integration of the derived density function, as is usual in the literature (see e.g. [2]). Thus we avoid using asymptotic expansions (as in [5]); however, some difficulties arise during the numerical integration (see e.g. [1]). They are caused by the oscillating nature of the so called Hartman-Watson distribution density function which is a part of the density function derived by Matsumoto and Yor [14].

3.2. **Closed form of the density function in the case of stochastic volatility being a 3-dimensional Bessel process.** We now consider another example of a linear stochastic volatility model. We assume that the process Y, representing the volatility of the financial asset, is a 3-dimensional Bessel process with $Y_0 = 1$. It is well known (see, e.g., Revuz and Yor [17]) that Y is a strong solution of the SDE

$$dY_t = dZ_t + \frac{1}{Y_t}dt, \tag{20}$$

where Z is a one-dimensional standard Wiener process. Since $Y_0 = 1$ and for any $t \le T$,

$$\mathbb{E}\int_0^t Y_u^2 du = \int_0^t (1+3u)du = t + \frac{3}{2}t^2 < \infty, \tag{21}$$

the SDE's (1) and (20) define a linear stochastic volatility model (with $\mu(y) = 1/y$ and $\sigma(\mu) = 1$ for $y > 0$). Moreover, it is easy to check, using the Itô lemma, that condition (4) is satisfied:

$$\mathbb{E}\int_0^t \frac{1}{Y_u^2}\, du = \mathbb{E}\ln Y_t^2 \le \mathbb{E}Y_t^2 = 1 + 3t < \infty. \tag{22}$$

Therefore, as we noticed in Remark 2.3, the problem of finding the density function of the asset reduces to deriving the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$. We do this in the next theorem finding the Laplace transform of this vector.

Theorem 3.3. *Let Y be a BES(3) process with $Y_0 = 1$. The Laplace transform of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$ is given by the formula*

$$\mathbb{E}\exp\left(-\lambda\int_0^t Y_u dZ_u - \frac{b^2}{2}\int_0^t Y_u^2 du\right) = \int_0^\infty p(t,y)y\exp\left(-\lambda\frac{t-1+y^2}{2}\right)H_{b,t}(y)dy, \tag{23}$$

where $\lambda > 0$, $b > 0$,

$$p(t,y) = \frac{1}{\sqrt{2\pi t}}\Big[\exp\left(-\frac{(y-1)^2}{2t}\right) - \exp\left(-\frac{(y+1)^2}{2t}\right)\Big],$$

and the function $H_{b,t}$ is defined for $y>0$ by the formula

$$H_{b,t}(y) = \sqrt{\frac{bt}{\sinh(bt)}}\exp\Bigg(-\frac{1}{2t}\Bigg\{(y^2[bt\coth(bt)-1]+2bt(y+1)\frac{\cosh(bt)-1}{\sinh(bt)}\Bigg\}\Bigg). \tag{24}$$

Proof. Note that the Laplace transform is well defined, since $\int_0^t Y_u^2du>0$ and

$$\int_0^t Y_u dZ_u = \frac{1}{2}[Y_t^2-1-3t] > -\frac{1+3T}{2}. \tag{25}$$

Take a probability space $(\Omega,\mathcal{F},\mathbb{Q})$ and a process Y which is, under $\mathbb{Q}$, a Brownian motion starting from 1. Define $\tau_0=\inf\{t\geq 0: Y_t=0\}$ and let $\mathbb{P}$ be a new probability measure given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = Y_{t\wedge\tau_0}.$$

Then, by the Girsanov theorem, Y_t is BES(3) with $Y_0=1$ under $\mathbb{P}$ (see Pal and Protter [15]). Fix $\lambda>0$ and $b>0$. Then, by (20) and the fact that $Y_{t\wedge\tau_0}=0$ for $t>\tau_0$,

$$\begin{aligned}\mathbb{E}\Bigg(&\exp\Bigg(-\lambda\int_0^t Y_u dZ_u - \frac{b^2}{2}\int_0^t Y_u^2 du\Bigg)\Bigg)\\ &= \mathbb{E}_{\mathbb{Q}}\Bigg[Y_{t\wedge\tau_0}\exp\Bigg(-\lambda\int_0^t Y_u[dY_u - Y_{u\wedge\tau_0}^{-1}du] - \frac{b^2}{2}\int_0^t Y_u^2du\Bigg)\Bigg].\end{aligned} \tag{26}$$

Let B be, under the measure $\mathbb{P}$, a Brownian motion starting from 1, and set $\tau_0^B = \inf\{t\geq 0: B_t=0\}$. Then (26) implies

$$\begin{aligned}\mathbb{E}\Bigg(&\exp\Bigg(-\lambda\int_0^t Y_u dZ_u - \frac{b^2}{2}\int_0^t Y_u^2du\Bigg)\Bigg)\\ &= \mathbb{E}\Bigg[B_{t\wedge\tau_0^B}\exp\Bigg(-\lambda\int_0^t B_u dB_u + \lambda t - \frac{b^2}{2}\int_0^t B_u^2 du\Bigg)\Bigg].\end{aligned}$$

If $\tau_0^B<t$, then the expression under expectation vanishes as $B_{\tau_0^B}=0$. Hence we can replace t by $t\wedge\tau_0^B$ under expectation, so

$$\begin{aligned}&\mathbb{E}\Bigg[B_{t\wedge\tau_0^B}\exp\Bigg(-\lambda\int_0^t B_u dB_u+\lambda t-\frac{b^2}{2}\int_0^t B_u^2du\Bigg)\Bigg]\\ &\quad= \mathbb{E}\Bigg[B_{t\wedge\tau_0^B}\exp\Bigg(-\lambda\frac{t\wedge\tau_0^B-1+B^2_{t\wedge\tau_0^B}}{2}-\frac{b^2}{2}\int_0^{t\wedge\tau_0^B}B_u^2du\Bigg)\Bigg]\\ &\quad= \mathbb{E}\Bigg[B_{t\wedge\tau_0^B}\exp\Bigg(-\lambda\frac{t\wedge\tau_0^B-1+B^2_{t\wedge\tau_0^B}}{2}\Bigg)\mathbb{E}\Bigg[\exp\Bigg(-\frac{b^2}{2}\int_0^{t\wedge\tau_0^B}B_u^2du\Bigg)\Bigg|B_{t\wedge\tau_0^B}\Bigg]\Bigg].\end{aligned} \tag{27}$$

From Theorem 5.1 (see Appendix) for $x=1$, it follows that

$$\mathbb{E}\Bigg[\exp\Bigg(-\frac{b^2}{2}\int_0^t B_u^2du\Bigg)\Bigg|B_t=y\Bigg] = H_{b,t}(y), \tag{28}$$

where $H_{b,t}$ is given by (24). Inserting (28) into (27) we obtain

$$\begin{aligned}
&\mathbb{E}\bigg(\exp\Big(-\lambda \int_0^t Y_u dZ_u - \frac{b^2}{2}\int_0^t Y_u^2 du\Big)\bigg) \\
&= \mathbb{E}\bigg[1_{\{B_{t\wedge\tau_0^B}>0\}} B_{t\wedge\tau_0^B} \exp\Big(-\lambda \frac{t\wedge\tau_0^B - 1 + B^2_{t\wedge\tau_0^B}}{2}\Big) H_{b,t\wedge\tau_0^B}(B_{t\wedge\tau_0^B})\bigg] \\
&= \mathbb{E}\bigg[1_{\{\tau_0^B>t\}} 1_{\{B_t>0\}} B_t \exp\Big(-\lambda \frac{t-1+B_t^2}{2}\Big) H_{b,t}(B_t)\bigg] \\
&= \int_0^\infty p(t,1,y) y \exp\Big(-\lambda \frac{t-1+y^2}{2}\Big) H_{b,t}(y) dy,
\end{aligned}$$

where

$$p(t,1,y) = \frac{1}{\sqrt{2\pi t}}\bigg[\exp\Big(-\frac{(y-1)^2}{2t}\Big) - \exp\Big(-\frac{(y+1)^2}{2t}\Big)\bigg]$$

is the transition function of Brownian motion starting at 1 and absorbed at zero (see Karatzas and Shreve [9, p. 97]). This concludes the proof. □

The last theorem provides the Laplace transform of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$, so we are able to invert it and obtain the density function in a linear stochastic volatility model in the case of the stochastic volatility being a 3-dimensional Bessel process. However, as shown below, we can solve this problem in another way. The difference between the two methods of finding the density function is that in the second one we do not need to invert the Laplace transform of a vector but of a function of one variable.

We start from a lemma computing $\mathbb{E}F(Y_t^2, \int_0^t Y_u^2 du)$ for a measurable function F and for Y being BES(3).

Lemma 3.4. *Let Y_t be a BES(3) process starting from 1, and g be the density function of the vector $(B_t, \int_0^t B_u^2 du)$, where B_t is a standard one-dimensional Brownian motion. Then for a measurable function F such that $\mathbb{E}F(Y_t^2, \int_0^t Y_u^2 du) < \infty$, we have*

$$\begin{aligned}
&\mathbb{E}F\Big(Y_t^2, \int_0^t Y_u^2 du\Big) = \\
&\int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty Q^*(x_1,y_1,x_2,y_2,x_3,y_3) dx_1 dy_1 dx_2 dy_2 dx_3 dy_3,
\end{aligned}$$

where

$$Q^*(x_1,y_1,x_2,y_2,x_3,y_3) = F(1+x^*, t+y^*) g(x_1,y_1) g(x_2,y_2) g(x_3,y_3),$$

and $x^* = x_1^2 + x_2^2 + x_3^2$, $y^* = y_1 + y_2 + y_3$.

Proof. If Y_t is a BES(3) process starting from 1, then

$$Y_t^2 = 1 + Z_{1,t}^2 + Z_{2,t}^2 + Z_{3,t}^2,$$

where $Z_{1,t}, Z_{2,t}, Z_{3,t}$ are independent standard Brownian motions (see [9]). The conclusion of the lemma now follows from the chain rule for conditional expectation:

$$\begin{aligned}
&\mathbb{E}F\Big(Y_t^2, \int_0^t Y_u^2 du\Big) = \mathbb{E}F\Big(1+Z_{1,t}^2+Z_{2,t}^2+Z_{3,t}^2,\ t+\int_0^t [Z_{1,u}^2+Z_{2,u}^2+Z_{3,u}^2]du\Big)\\
&= \mathbb{E}\mathbb{E}\Big[F\Big(1+Z_{1,t}^2+Z_{2,t}^2+Z_{3,t}^2,\ t+\int_0^t [Z_{1,u}^2+Z_{2,u}^2+Z_{3,u}^2]du\Big)|\mathcal{F}_t^{Z_1,Z_2}\Big]\\
&= \int_{-\infty}^{\infty}\int_0^{\infty} g(x_1,y_1)\mathbb{E}F\Big(1+x_1^2+Z_{2,t}^2+Z_{3,t}^2,\ t+y_1+\int_0^t [Z_{2,u}^2+Z_{3,u}^2]du\Big)dy_1dx_1\\
&= \ldots
\end{aligned}$$

□

Using this lemma we find the density function of the asset price X_t.

Theorem 3.5. *Let Y_t be a BES(3) process with $Y_0 = 1$, and g be the density function of the vector $(B_t, \int_0^t B_u^2 du)$, where B_t is a standard one-dimensional Brownian motion. Then the density function of X_t, for fixed $t \geq 0$, is given by*

$$g_{X_t}(r) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} Q(r, X_0, x_1, y_1, x_2, y_2, x_3, y_3)dx_1dy_1dx_2dy_2dx_3dy_3,$$

with

$$Q(r, X_0, x_1, y_1, x_2, y_2, x_3, y_3) = \Phi'\Big(\frac{\ln\frac{r}{X_0} - \frac{\rho}{2}[x^* - 3t] - \frac{1}{2}[t+y^*]}{\sqrt{(1-\rho^2)[t+y^*]}}\Big)\frac{g(x_1,y_1)g(x_2,y_2)g(x_3,y_3)}{r\sqrt{(1-\rho^2)(t+y^*)}},$$

where $x^* = x_1^2 + x_2^2 + x_3^2$, $y^* = y_1 + y_2 + y_3$ and Φ is the cumulative distribution function of a standard Gaussian random variable $N(0, 1)$.

Proof. Since (22) is satisfied, we conclude from Theorem 2.2 that

$$g_{X_t}(r) = \mathbb{E}\Big[\frac{1}{r\sqrt{(1-\rho^2)\int_0^t Y_u^2du}}\Phi'\Big(\frac{\ln\frac{r}{X_0} - \frac{\rho}{2}[Y_t^2 - 1 - 3t] - \frac{1}{2}\int_0^t Y_u^2du}{\sqrt{(1-\rho^2)\int_0^t Y_u^2du}}\Big)\Big],$$

and the assertion follows immediately from Lemma 3.4. □

Remark 3.6. In order to complete the example of BES(3) volatility (starting from 1) we comment on the density function g of the vector $(B_t, \int_0^t B_u^2 du)$, where B_t is a standard Brownian motion. It is proved in the Appendix (see Corollary 5.2) that for $c > 0$, $t \geq 0$, $x \in \mathbb{R}$,

$$\int_0^{\infty} e^{-cy} g(x,y)dy = H^*(t,c,x), \tag{29}$$

where

$$H^*(t,c,x) := \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{2c}}{\sinh(t\sqrt{2c})}}\exp\Big(-\frac{x^2}{2}\sqrt{2c}\coth(t\sqrt{2c})\Big). \tag{30}$$

So if we denote by $\mathcal{L}^{-1}$ the inverse Laplace transform, then

$$g = \mathcal{L}^{-1}H^* \tag{31}$$

and we obtain the density function of the vector $(B_t, \int_0^t B_u^2 du)$.

4. Closed form of the arbitrage price of a vanilla option in a linear stochastic volatility model

In this section we derive a probabilistic representation of a vanilla option price in a linear stochastic volatility model. We are interested in computation of the arbitrage prices, so the process X describing the discounted price of the asset should be a martingale. Next, as examples, we show how to deduce from Theorem 4.1 closed formulae for option prices for the models of Section 3. In our examples we give conditions guaranteeing that X is a martingale. Then, just as in Section 2, we show how the valuation of vanilla options in that model can be reduced to finding the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$.

4.1. **Probabilistic representation of the arbitrage price of a vanilla option in a linear stochastic volatility model.** Under the assumptions of Theorem 2.2 and for the process X being a martingale, we provide probabilistic representations for the arbitrage prices of European call and put options. These formulae generalize the famous Black-Scholes formulae as well as the result of Hull and White for a stochastic volatility model with uncorrelated noises [6].

Theorem 4.1. *Assume that X is a martingale. In a linear stochastic volatility model, under assumption (4), the time zero arbitrage prices of European call and put options with strike $K>0$ and maturity t have the following probabilistic representation:*

$$\begin{aligned}(32)\quad &\mathbb{E}[X_t - K]^+ \\ &= X_0\mathbb{E}\Big[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Phi\Big(\frac{\ln\frac{X_0}{K}+\mu_Z(t)+\sigma_Z^2(t)}{\sigma_Z(t)}\Big)\Big] - K\mathbb{E}\Phi\Big(\frac{\ln\frac{X_0}{K}+\mu_Z(t)}{\sigma_Z(t)}\Big),\end{aligned}$$

$$\begin{aligned}(33)\quad &\mathbb{E}[K - X_t]^+ \\ &= K\mathbb{E}\Phi\Big(\frac{\ln\frac{K}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big) - X_0\mathbb{E}\Big[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Phi\Big(\frac{\ln\frac{K}{X_0}-\mu_Z(t)-\sigma_Z^2(t)}{\sigma_Z(t)}\Big)\Big],\end{aligned}$$

where $\mu_Z(t)$ and $\sigma_Z^2(t)$ are given by (6) and (7).

Proof. Let $K>0$. We compute $\mathbb{E}[K-X_t]^+$. Integrating by parts, using the probabilistic representation of g_{X_t}, i.e. (5), we have

$$\begin{aligned}\mathbb{E}[K-X_t]^+ &= \int_0^\infty (K-r)^+ g_{X_t}(r)dr = \int_0^\infty (K-r)^+\frac{\partial}{\partial r}\mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)dr \\ &= (K-r)\mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)\Big|_0^K + \int_0^K \mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)dr \\ &= \int_0^K \mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)dr,\end{aligned}$$

where in the last expression we have used (12). Next, by the Fubini theorem,

$$\int_0^K \mathbb{E}\Phi\Big(\frac{\ln\frac{r}{X_0}-\mu_Z(t)}{\sigma_Z(t)}\Big)dr = \mathbb{E}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}\Big[\int_0^K 1_{\{r\geq X_0 e^{s\sigma_Z(t)+\mu_Z(t)}\}}dr\Big]ds = I_3,$$

where

$$I_3 = \mathbb{E}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\Big[K - X_0 e^{s\sigma_Z(t)+\mu_Z(t)}\Big]^+ ds.$$

Let us define

$$K^* = \frac{K}{X_0} e^{-\mu_Z(t)}, \quad d_1^* = \frac{\ln\frac{X_0}{K} + \mu_Z(t) + \sigma_Z^2(t)}{\sigma_Z(t)}, \quad d_2^* = \frac{\ln\frac{X_0}{K} + \mu_Z(t)}{\sigma_Z(t)}.$$

Then, changing variable s to $v = s\sigma_Z(t)$ in I_3, we obtain (33):

$$\begin{aligned} I_3 &= \mathbb{E}\bigg(X_0 e^{\mu_Z(t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z(t)} e^{-v^2/2\sigma_Z^2(t)}\Big[K^* - e^v\Big]^+ dv\bigg) \\ &= \mathbb{E}\bigg(X_0 e^{\mu_Z(t)}\bigg[K^*\Phi\Big(\frac{\ln K^*}{\sigma_Z(t)}\Big) - e^{\sigma_Z^2(t)/2}\Phi\Big(\frac{\ln K^* - \sigma_Z^2(t)}{\sigma_Z(t)}\Big)\bigg]\bigg) \\ &= K\mathbb{E}\Phi(-d_2^*) - X_0\mathbb{E}[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Phi(-d_1)]. \end{aligned}$$

Hence, as X is by assumption a martingale,

$$\begin{aligned} \mathbb{E}[X_t - K]^+ &= X_0 - K - \mathbb{E}[K - X_t]^+ \\ &= X_0(1 - \mathbb{E}[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Phi(-d_1^*)]) - K(1 - \mathbb{E}\Phi(-d_2^*)). \end{aligned} \tag{34}$$

Let us observe that

$$e^{\mu_Z(t)+\sigma_Z^2(t)/2} = e^{\rho\int_0^t Y_u dZ_u - \rho^2\int_0^t Y_u^2 du/2}.$$

Moreover,

$$\begin{aligned} \mathbb{E}e^{\int_0^t Y_u dW_u - \int_0^t Y_u^2 du/2} &= \mathbb{E}\mathbb{E}\Big[e^{\rho\int_0^t Y_u dZ_u + \sqrt{1-\rho^2}\int_0^t Y_u dB_u - \int_0^t Y_u^2 du/2} | \mathcal{F}_t^Z\Big] \\ = \mathbb{E}\Big[e^{\rho\int_0^t Y_u dZ_u - \int_0^t Y_u^2 du/2}\mathbb{E}\Big[e^{\sqrt{1-\rho^2}\int_0^t Y_u dB_u} | \mathcal{F}_t^Z\Big]\Big] &= \mathbb{E}e^{\rho\int_0^t Y_u dZ_u - \rho^2\int_0^t Y_u^2 du/2}. \end{aligned} \tag{35}$$

Since X given by (1) is a martingale, we have $1 = \mathbb{E}e^{\int_0^t Y_u dW_u - \int_0^t Y_u^2 du/2}$ and so the process $(e^{\mu_Z(t)+\sigma_Z^2(t)/2})_t$ is a martingale. Hence by (34) we finally conclude that

$$\begin{aligned} \mathbb{E}[X_t - K]^+ &= X_0\mathbb{E}\Big[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Big(1 - \Phi(-d_1^*)\Big)\Big] - K\mathbb{E}\Big(1 - \Phi(-d_2^*)\Big) \\ &= X_0\mathbb{E}\Big[e^{\mu_Z(t)+\sigma_Z^2(t)/2}\Phi(d_1^*)\Big] - K\mathbb{E}\Phi(d_2^*), \end{aligned}$$

which ends the proof. □

4.2. **Examples.** In this subsection we consider the previously discussed models.

4.2.1. *Black-Scholes and stochastic log-normal volatility models.* In these two cases, closed formulae for the arbitrage price of European call and put options with strike $K > 0$ can be derived. We emphasize that these results are not a direct consequence of deriving the density function for the model. Rather, they are consequences of the probabilistic representation (see Theorem 4.1) of the arbitrage price of vanilla option in a linear stochastic volatility model.

In the case of the Black-Scholes model, $\mu_Z(t) = -t\sigma^2/2$ and $\sigma_Z^2(t) = \sigma^2 t$, so (32) and (33) immediately give the famous Black-Scholes formulae.

As before, the case of a log-normal stochastic volatility model is less trivial. We give formulae for the arbitrage prices of vanilla options in such models (different formulae were obtained in [12] in another way).

Remark 4.2. Jourdain [8] proved that the condition $\rho \in (-1, 0]$ is equivalent to X being a martingale. So, in further considerations, whenever we need X to be martingale, we consider only nonpositive ρ, and in this case $\mathbb{P}$ is a martingale measure.

Theorem 4.3. *In a log-normal stochastic volatility model with $\rho \le 0$, under assumption* (4)*, the time zero arbitrage prices of European call and put options with strike $K > 0$ and maturity t are given by*

$$\begin{aligned}(36)\quad & \mathbb{E}[X_t - K]^+ \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \Big[X_0 e^{f(x,y)}\Phi(d_1(x,y)) - K\Phi(d_2(x,y))\Big] G_{t\sigma^2}(x,y)\,dy\,dx,\end{aligned}$$

$$\begin{aligned}(37)\quad & \mathbb{E}[K - X_t]^+ \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \Big[K\Phi(-d_2(x,y)) - X_0 e^{f(x,y)}\Phi(-d_1(x,y))\Big] G_{t\sigma^2}(x,y)\,dy\,dx,\end{aligned}$$

where f, G are given by (15) *and* (16)*, and*

$$d_1(x,y) = \frac{\ln\frac{X_0}{K} + f(x,y)}{Y_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}} + \frac{Y_0}{2}\sqrt{\frac{1-\rho^2}{\sigma^2}y},$$

$$d_2(x,y) = d_1(x,y) - \frac{Y_0}{2}\sqrt{\frac{1-\rho^2}{\sigma^2}y}.$$

Proof. The assumption $\rho \le 0$ and Remark 4.2 imply that X is a martingale. Arguing as in the proof of Theorem 3.1 and using the same notation we have, by Theorem 4.1,

$$(38)\qquad \mathbb{E}[X_{\frac{t}{\sigma^2}} - K]^+ = \mathbb{E}\big[X_0 e^{f(V_t,A_t)}\Phi(d_1(V_t,A_t)) - K\Phi(d_2(V_t,A_t))\big],$$

$$(39)\qquad \mathbb{E}[K - X_{\frac{t}{\sigma^2}}]^+ = \mathbb{E}\big[-K\Phi(-d_2(V_t,A_t)) - X_0 e^{f(V_t,A_t)}\Phi(-d_1(V_t,A_t))\big],$$

and hence

(40)

$$\mathbb{E}[X_{\frac{t}{\sigma^2}} - K]^+ = \int_{-\infty}^{\infty}\int_0^{\infty} \Big[X_0 e^{f(x,y)}\Phi(d_1(x,y)) - K\Phi(d_2(x,y))\Big] G_t(x,y)\,dy\,dx,$$

(41)

$$\mathbb{E}[K - X_{\frac{t}{\sigma^2}}]^+ = \int_{-\infty}^{\infty}\int_0^{\infty} \Big[K\Phi(-d_2(x,y)) - X_0 e^{f(x,y)}\Phi(-d_1(x,y))\Big] G_t(x,y)\,dy\,dx.$$

To conclude the proof we replace t by $t\sigma^2$ in (40) and (41). □

4.2.2. *Model with BES(3) volatility starting from 1.* In this case we formulate a condition under which the probabilistic representations hold.

Proposition 4.4. *Assume $\rho \le 0$. If Y is a BES(3) process starting from 1, then X_t is a martingale and the probabilistic representations of the arbitrage prices of vanilla call and put options hold.*

Proof. (21) implies that $\int_0^t Y_u dW_u$ is a martingale. Using (3), representation (8) and (35) we obtain

$$\mathbb{E}X_t = X_0 \mathbb{E}\Big[e^{\rho\int_0^t Y_u dZ_u - \rho^2 \int_0^t Y_u^2 du/2}\Big].$$

Formula (25) and $\rho \le 0$ imply that the local martingale under the last expectation is bounded by $X_0 e^{-\rho(3T+1)/2}$, so it is a martingale and this expectation is equal to one. Therefore $EX_t = X_0$ and X is a martingale. The assertion of the theorem now follows from Theorem 4.1. $\square$

5. Appendix

In this appendix we provide some interesting identities for the Laplace transform of $(B_t, \int_0^t B_u^2 du)$ for a Brownian motion B starting from any point $x \in \mathbb{R}$. The first part of Theorem 5.1 is well-known (see Mansuy and Yor [13, p.18]). We give here a short proof for completeness. The second part is a conditional version of the Donati-Martin and Yor formula [3]. Finally, Corollary 5.2 establishes the density function of the vector $(B_t, \int_0^t B_u^2 du)$ for a standard Brownian motion B.

Theorem 5.1. *Let B_t be a Brownian motion starting from $x \in \mathbb{R}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any $c > 0, b > 0$ and $y \in \mathbb{R}$:*

$$\begin{aligned}
(42)\quad & \mathbb{E}\bigg(\exp\Big(-cB_t^2 - \frac{b^2}{2}\int_0^t B_u^2 du\Big)\bigg) \\
&= \frac{1}{\sqrt{\cosh(bt) + \frac{2c}{b}\sinh(bt)}} \exp\bigg(x^2\Big[\frac{b}{2} - \frac{(b/2+c)e^{bt}}{\cosh(bt) + \frac{2c}{b}\sinh(bt)}\Big]\bigg), \\
(43)\quad & \mathbb{E}\Big[\exp\Big(-\frac{b^2}{2}\int_0^t B_u^2 du\Big)\Big| B_t = y\Big] = \\
& \sqrt{\frac{bt}{\sinh(bt)}} \exp\bigg(-\frac{1}{2t}\Big[y^2(bt\coth(bt) - 1) + 2btx(x+y)\frac{\cosh(bt)-1}{\sinh(bt)}\Big]\bigg).
\end{aligned}$$

Proof. Observe that $B_t = V_t + x$, where V_t is a standard Brownian motion under $\mathbb{P}$. Let $\mathbb{P}^*$ be a new probability measure defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\bigg(b\int_0^t (V_u + x)dV_u - \frac{b^2}{2}\int_0^t (V_u + x)^2 du\bigg). \tag{44}$$

Then, by Girsanov's theorem, $V_t^* = V_t - b\int_0^t (V_s + x)ds$ is a standard Brownian motion under $\mathbb{P}^*$. Moreover, $V_t = x[e^{bt} - 1] + e^{bt}\int_0^t e^{-bs} dV_s^*$ (see, e.g., Karatzas

and Shreve [9]). Consequently, changing measure we deduce

$$
\begin{aligned}
(45)\qquad & \mathbb{E}\bigg(\exp\Big(-cB_t^2-\frac{b^2}{2}\int_0^t B_u^2du\Big)\bigg)\\
&=\mathbb{E}_{\mathbb{P}^*}\bigg(\exp\Big(-c(V_t+x)^2-b\int_0^t(V_u+x)dV_u\Big)\bigg)\\
&=\mathbb{E}_{\mathbb{P}^*}\bigg(\exp\Big(-c(V_t+x)^2-b\frac{(V_t+x)^2-t-x^2}{2}\Big)\bigg)\\
&=\mathbb{E}_{\mathbb{P}^*}\exp\Big(\frac{b(t+x^2)}{2}-(b/2+c)e^{2bt}(x+\int_0^t e^{-bs}dV_s^*)^2\Big)\\
&=\exp\Big(\frac{b(t+x^2)}{2}\Big)\mathbb{E}_{\mathbb{P}^*}\exp\Big(-(b/2+c)e^{2bt}(x+\int_0^t e^{-bs}dV_s^*)^2\Big)\\
&=\frac{1}{\sqrt{\cosh(bt)+\frac{2c}{b}\sinh(bt)}}\exp\bigg(x^2\Big[\frac{b}{2}-\frac{(b/2+c)e^{bt}}{\cosh(bt)+\frac{2c}{b}\sinh(bt)}\Big]\bigg),
\end{aligned}
$$

as $\int_0^t e^{-bs}dV_s^*$ is a Gaussian random variable (with respect to $\mathbb{P}^*$) with mean 0 and variance $\frac{1}{2b}[1-e^{-2bt}]$, and for a Gaussian random variable U with mean μ and variance σ^2, for any $\alpha>0$, we have

$$
\mathbb{E}_{\mathbb{P}^*}e^{-\alpha U^2}=\frac{1}{\sqrt{2\alpha\sigma^2+1}}\exp\Big(-\frac{\alpha\mu^2}{2\alpha\sigma^2+1}\Big).
$$

So (42) is proved. After some algebra we obtain

$$
\begin{aligned}
(46)\qquad & \mathbb{E}\bigg(\exp\Big(-cB_t^2-\frac{b^2}{2}\int_0^t B_u^2du\Big)\bigg)\\
&=\frac{1}{\sqrt{\cosh(bt)+\frac{2c}{b}\sinh(bt)}}\exp\Big(-cx^2+x^2\frac{\frac{2}{b}(c^2-b^2/4)}{\coth(bt)+2c/b}\Big).
\end{aligned}
$$

We now prove the second identity. Let us define, for $t\geq 0$,

$$
(47)\qquad H_{b,t,x}(y):=\mathbb{E}\bigg[\exp\Big(-\frac{b^2}{2}\int_0^t B_u^2du\Big)\Big|B_t=y\bigg].
$$

It is clear that for $t=0$ we have $H_{b,0,x}(y)=1$ for any $y\in\mathbb{R}$ and $0<H_{b,t,x}\leq 1$. The function $H_{b,t,x}(\cdot)$ must satisfy, for all $c>0$,

$$
(48)\qquad \mathbb{E}\bigg(\exp\Big(-cB_t^2-\frac{b^2}{2}\int_0^t B_u^2du\Big)\bigg)=\mathbb{E}\Big[e^{-cB_t^2}H_{b,t,x}(B_t)\Big].
$$

It is easy to see that there can only be one such function $H_{b,t,x}$. Let us try to find $H_{b,t,x}$ in the form

$$
\begin{aligned}
(49)\qquad & H_{b,t,x}(y)=\\
& F(b,t,x)\exp\bigg(-\frac{1}{2t}\Big[(K(b,t,x)-1)(y-x)^2+2L(b,t,x)(y-x)+R(b,t,x)\Big]\bigg)
\end{aligned}
$$

for some functions K,L,R,F of three variables b,t,x. To ease calculations we omit the arguments of the functions K,L,R,F. Recall that $B_t=V_t+x$, where V_t is a

standard Brownian motion under $\mathbb{P}$, so

$$\begin{aligned}\mathbb{E}\Big[e^{-cB_t^2}H_{b,t,x}(B_t)\Big] &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}F\exp\bigg(-c(x+y)^2-\frac{1}{2t}[Ky^2+2Ly+R]\bigg)dy\\ &= F\frac{1}{\sqrt{K+2tc}}\exp\bigg(-cx^2+\frac{1}{2t}\bigg[\frac{(L+2ctx)^2}{K+2ct}-R\bigg]\bigg).\end{aligned}$$

We now match the last expression with (46) and guess that

$$\begin{aligned}F&=F(b,t,x)=F(b,t)=\sqrt{\frac{bt}{\sinh(bt)}},\\ K&=K(b,t,x)=K(b,t)=bt\coth(bt),\\ R&=2Lx,\\ 0&=L^2-2LKx+b^2x^2t^2.\end{aligned}$$

The last equation has two solutions $L_1 = xbt(\cosh(bt)+1)/\sinh(bt)$ and $L_2 = xbt(\cosh(bt)-1)/\sinh(bt)$ but the first one tends to infinity when t tends to zero (causing H to be 0 in the limit). But from the Lebesgue theorem we clearly see that

$$\lim_{t\to 0}\mathbb{E}H_{b,t,x}(B_t)=\lim_{t\to 0}\mathbb{E}\bigg(\exp\bigg(-\frac{b^2}{2}\int_0^t B_u^2du\bigg)\bigg)=1,$$

so what is left is the second solution that fits perfectly our computation. Hence, with

$$\begin{aligned}L&=L(b,t,x)=xbt\frac{\cosh(bt)-1}{\sinh(bt)}\\ R&=R(b,t,x)=2xB(b,t,x)=2x^2bt\frac{\cosh(bt)-1}{\sinh(bt)},\end{aligned}$$

and F, K defined above, the function $H_{b,t,x}$ satisfies the condition (48) by construction. To conclude the proof we insert F, K, L, R just computed in (49) and obtain the assertion of the theorem. □

Corollary 5.2. *Fix $t \geq 0$. Let g be the density function of the vector $(B_t, \int_0^t B_u^2 du)$, where B_t is a standard Brownian motion. Then for $z \in \mathbb{R}$ and $c > 0$,*

$$\int_0^\infty e^{-cy}g(z,y)dy = H^*(t,c,z), \tag{50}$$

where

$$H^*(t,c,z) := \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{2c}}{\sinh(t\sqrt{2c})}}\exp\bigg(-\frac{z^2}{2}\sqrt{2c}\coth(t\sqrt{2c})\bigg). \tag{51}$$

Proof. This is an immediate consequence of formula (43) with $x = 0$, $b = \sqrt{2c}$ and the fact that

$$\int_0^\infty e^{-cy}g(z,y)dy = H_{\sqrt{2c},t,0}(z)g_{B_t}(z), \tag{52}$$

where $g_{B_t}(z) = \frac{1}{\sqrt{2\pi t}}e^{-z^2/2t}$ denotes the density function of the random variable B_t, and $H_{b,t,x}$ is defined by (47).

□

References

[1] Barrieu P., Rouault A., Yor M. *A study of the Hartman-Watson distribution motivated by numerical problems related to the pricing of Asian options.* J. Appl. Probab. 41, No. 4, 1049-1058 (2004).

[2] Carmona R., Durrleman V. *Pricing and hedging spread options.* SIAM Rev. 45, No. 4, 627-685 (2003).

[3] Donati-Martin C., Yor M. *Some Brownian functionals and their laws.* Ann. Probab. 25, 1011-1058 (1997).

[4] Forde M. *Tail asymptotics for diffusion processes, with applications to local volatility and CEV-Heston models.* Working paper (2008).

[5] Hagan P., Kumar D., Lesniewski A., Woodward D. *Managing smile risk.* Wilmott Magazine, September, 84-108 (2002).

[6] Hull, J., White, A. *The pricing of options on assets with stochastic volatilities.* J. Finance 42, 281-300 (1987).

[7] Ikeda N., Watanabe S. *Stochastic Differential Equations and Diffusion Processes.* North-Holland Kodansha 1981.

[8] Jourdain B. *Loss of martingality in asset price model with log-normal stochastic volatility.* ENPC-CERMICS, Working paper (2004).

[9] Karatzas I., Shreve S. *Brownian Motion and Stochastic Calculus.* Springer-Verlag 1991).

[10] Leblanc B. *Une approche unifiée pour une forme exacte du prix d'une option dans les différents modèles à volatilité stochastique.* Stochastics and Stochastics Reports 57, 1-35 (1996).

[11] Maghsoodi Y. *Exact solution of a martingale stochastic volatility option problem and its empirical evaluation.* Mathematical Finance 17, No. 2, 249-265 (2007).

[12] Maghsoodi Y. *Exact solution of the log-normal stochastic volatility option problem and its empirical evaluation.* Preprint (2007).

[13] Mansuy R., Yor M. *Aspects of Brownian Motion.* Universitext, Springer-Verlag 2008.

[14] Matsumoto H., Yor M. *Exponential functionals of Brownian motion, I, Probability laws at fixed time.* Probab. Surveys 2, 312-347 (2005).

[15] Pal S., Protter P. *Strict local martingales, bubbles and no early exercise.* Preprint (2007).

[16] Rebonato R. *Volatility and Correlation. The Perfect Hedger and the Fox.* Wiley (2nd ed.) 2004.

[17] Revuz D., Yor M. *Continous Martingales and Brownian Motion.* Springer-Verlag (3rd ed.). 2005.

[18] Romano M., Touzi N. *Contingent claims and market completeness in a stochastic volatility model.* Mathematical Finance 7, No. 4, 399-410 (1997).

[19] Wystup U. *FX Options and Structured Products.* Wiley. 2006.

[20] Yor M. *On some exponential functionals of Brownian motion.* Adv. Appl. Probab. 24, 509-531 (1992).